



**HAL**  
open science

# **Kinematics of nonlinear waves over variable bathymetry. Part I: Numerical modelling, verification and validation**

Michel Benoit, Jie Zhang, Yuxiang Ma

## ► To cite this version:

Michel Benoit, Jie Zhang, Yuxiang Ma. Kinematics of nonlinear waves over variable bathymetry. Part I: Numerical modelling, verification and validation. Coastal Engineering, 2024, 193, pp.104577. <10.1016/j.coastaleng.2024.104577>. <hal-04681767>

**HAL Id: hal-04681767**

**<https://edf.hal.science/hal-04681767v1>**

Submitted on 30 Aug 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# Kinematics of nonlinear waves over variable bathymetry. Part I: Numerical modelling, verification and validation

Michel Benoit<sup>a,b</sup>, Jie Zhang<sup>c,d,\*</sup> and Yuxiang Ma<sup>e</sup>

<sup>a</sup>EDF R&D, Laboratoire National d'Hydraulique et Environnement (LNHE), Chatou, France

<sup>b</sup>LHSV, Saint-Venant Hydraulics Laboratory (Ecole des Ponts, EDF R&D), Chatou, France

<sup>c</sup>Qingdao Innovation and Development Base, Harbin Engineering University, Qingdao 266400, PR China

<sup>d</sup>Qingdao Innovation and Development Center of Harbin Engineering University, Qingdao 266400, PR China

<sup>e</sup>State Key Laboratory of Coastal and Offshore Engineering, Dalian University of Technology, Dalian 116023, PR China

## ARTICLE INFO

### Keywords:

Coastal waves

Nonlinear waves

Wave kinematics

Numerical simulation

Statistical parameters

## ABSTRACT

Fluid particle kinematics due to wave motion (i.e. orbital velocities and accelerations) at and beneath the free surface is involved in many coastal and ocean engineering applications, e.g. estimation of wave-induced forces on structures, sediment transport, etc. This work presents the formulations of these kinematics fields within a fully nonlinear potential flow (FNPF) approach. In this model, the velocity potential is approximated with a high-order polynomial expansion over the water column using an orthogonal basis of Chebyshev polynomials of the first kind. Using the same basis, original analytical expressions of the components of velocity and acceleration are derived in this work. The estimation of particle accelerations in the course of the simulation involves the time derivatives of the decomposition coefficients, which are computed with a high-order backward finite-difference scheme in time. The capability of the numerical model in computing the particle kinematics is first validated for regular nonlinear waves propagating over a flat bottom. The model is shown to be able to predict both the velocity and acceleration of highly nonlinear and nearly breaking waves with negligible error compared to the corresponding stream function wave solution. Then, for regular waves propagating over an uneven bottom (bar-type bottom profile), the simulated results are confronted with existing experimental data, and very good agreement is achieved up to the sixth-order harmonics for free surface elevation, velocity and acceleration.

## 1. Introduction

Water wave kinematics has been studied for decades because it is of core importance in the design procedures of coastal and harbour engineering (e.g. Fredsøe and Deigaard, 1992). The distribution of total pressure, water particle velocities and accelerations plays a fundamental role when evaluating forces on marine structures, the motion of sediments and the evolution of coastal morphology (Freilich and Guza, 1984; Kriebel, 1998; Elfrink and Baldock, 2002; Wilson, 2002). Under the background of global warming, coastal and harbour engineering is faced with more frequent and disruptive extreme events (see e.g. Didenkulova and Pelinovsky, 2016, 2020; Didenkulova et al., 2023; Shi et al., 2024). To balance safety and economy, a better understanding and prediction of the flow field beneath strongly nonlinear waves remains of paramount importance (Zelt et al., 1995; Stansberg et al., 1995; Aggarwal et al., 2016; Vested et al., 2020; Li et al., 2023; Deng et al., 2023).

To obtain the spatial and temporal evolution of wave kinematics, deterministic (phase-resolving) models are needed. The Computational Fluid Dynamics (CFD) approach solves the Navier–Stokes equations which account for nonlinearity, vorticity and viscosity. It is very powerful in describing the flow of fluids, and in offering the pressure and velocity pro-

files. The Reynolds Averaged Navier–Stokes (RANS) equations, solved with either Lagrangian (Dalrymple and Rogers, 2006; Antuono et al., 2011) or Eulerian (Wang et al., 2009; Higuera et al., 2013; Jacobsen et al., 2015) methods, are often used in the studies of wave breaking, wave-structure interactions, and multiphase flows. One drawback of RANS equations is the high computational burden, limiting the spatial and temporal scale of the simulations. In addition, as shown recently by Larsen et al. (2019) for instance, the accuracy of computed wave kinematics is not always guaranteed with this kind of code, in particular in the vicinity of the free surface, depending on the employed numerical methods and the discretization parameters selected to solve this set of equations numerically.

Besides, there are very efficient yet simplified wave models with assumptions on the magnitudes of wave steepness and/or relative water depth, namely the weakly nonlinear and dispersive models. For instance, mild slope equation, Boussinesq equation, nonlinear Schrödinger equation, Serre-Green-Naghdi equation belong to this class (Berkhoff, 1972; Porter, 2003; Boussinesq, 1872; Madsen and Schäffer, 1998; Hasimoto and Ono, 1972; Dysthe, 1979; Green et al., 1974; Bassi et al., 2020, just to name a few in these topics). As the name indicates, models of this type are developed for handling weakly nonlinear and dispersive wave cases, i.e. in the long-wave regime. Therefore, they assume simplified representations of the profiles of horizontal and vertical velocity components in the vertical direction, typically using low-order polynomial expressions (i.e. linear or quadratic). Al-

\*Corresponding author

✉ michel.benoit@edf.fr (M. Benoit); jie.zhang@hrbeu.edu.cn (J. Zhang); yuxma@dlut.edu.cn (Y. Ma)

ORCID(s): 0000-0003-4195-2983 (M. Benoit); 0000-0003-0794-2335 (J. Zhang); 0000-0003-4314-0428 (Y. Ma)

though there are numerous extensions of the above-mentioned models, all being able to include higher-order effects, they often end up in cumbersome mathematical expressions with high-order derivatives which are tricky for numerical implementation.

As a compromise between efficiency and accuracy, given the viscous and turbulent effects are often negligible for wave propagation studies, the fully nonlinear potential flow (FNPF) model represents a powerful tool for wave modelling in a range of dozens of wavelengths. In recent years, it has received considerable attention and undergone substantial progress.

On the one hand, this mathematical model makes no a priori assumption about the magnitude of wave steepness and relative water depth, thus it is capable of describing all nonlinear features of non-overtopping waves. On the other hand, it is convenient to compute the particle velocity and acceleration from the scalar velocity potential. Besides, it can be used to provide input wave fields and to couple with CFD models (Paulsen et al., 2014; Decorte et al., 2021). The FNPF model can be solved with the Boundary Element Method (BEM) that projects the problem on the boundary surface of the fluid domain using Green's identity and function (Grilli et al., 1989; Fochesato and Dias, 2006; Harris et al., 2014) or with the so-called using "Zakharov formulation" that formulates the problem with free surface variables (Zakharov, 1968; Craig and Sulem, 1993), both methods allowing for a reduction of the dimension of the computational problem. Among many approaches, the High-Order Spectral method (Dommermuth, 2000; Gouin et al., 2016; Lawrence et al., 2021a), high-order Boussinesq-type models (Madsen et al., 2006; Bingham et al., 2009) and the Hamiltonian couple-mode theory (Belibassakis and Athanassoulis, 2011; Papoutsellis et al., 2019) have been used to solve the Zakharov formulation of the FNPF model. Recently, a numerical solver of the Zakharov formulation has been developed using a highly accurate approximation of the velocity potential through the use of a spectral approach in the vertical direction with a set of orthogonal polynomials (Tian and Sato, 2008; Yates and Benoit, 2015; Raoult et al., 2016; Benoit et al., 2017). The resulting computational code, called Whispers3D (abbreviated W3D hereafter), shows excellent accuracy in predicting the evolution of highly nonlinear free surface waves over steep slopes (Zhang and Benoit, 2021; Zhang et al., 2022). The W3D code is also capable of handling wave breaking with various options for breaking initiation criteria and dissipation mechanisms (Simon et al., 2019). However, the capability of this model in describing wave kinematics at and beneath the free surface elevation (FSE) has not been discussed yet, except partially in the work of Zhang and Benoit (2021). In the present work, W3D performance in calculating the velocity and acceleration directly and explicitly is thoroughly examined, namely by first deriving analytical expressions of these kinematic fields, and then applying these new results to compute velocity and acceleration fields for highly nonlinear waves over flat and uneven bottoms.

The remainder of this article is organized as follows: in section 2, the computation formulas of particle velocity and

acceleration are derived within the model formulation; section 3 presents the verification of the method for highly nonlinear regular waves case propagating in uniform water depth against analytical results computed with the exact stream function (SF) theory; section 4 shows the experimental validation for a nonlinear regular wave shoaling and de-shoaling case over a submerged trapezoidal bar, the simulation results are compared with measurements in both spectral and statistical domain. Conclusions are provided in section 5.

## 2. Mathematical wave model and internal kinematics

### 2.1. Overview of the wave model

Within the framework of potential theory, assuming that (i) the fluid is inviscid and incompressible, (ii) the flow is irrotational, and (iii) the waves are non-breaking, the fluid motion can be described by the velocity potential  $\phi$ . In addition, the free surface tension is neglected here, for simplicity. The above assumptions do not imply any restriction on the magnitude of the wave steepness nor on the relative water depth, thus the full nonlinearity and dispersion of waves are retained. Consider a two-dimensional case in a Cartesian coordinate system  $(x, z)$ , with  $z$ -axis pointing upward and  $z = 0$  located at the still water level (SWL), the governing equations of the FNPF theory are:

$$\Delta\phi = 0 \quad \text{for } z \in [-h, \eta], \quad (1)$$

$$\eta_t + \phi_x \eta_x - \phi_z = 0 \quad \text{on } z = \eta, \quad (2)$$

$$\phi_t + \frac{1}{2} (\phi_x^2 + \phi_z^2) + g\eta = 0 \quad \text{on } z = \eta, \quad (3)$$

$$h_x \phi_x + \phi_z = 0 \quad \text{on } z = -h, \quad (4)$$

where  $\Delta$  denotes the Laplace operator,  $\eta(x, t)$  denotes the FSE,  $h(x)$  the water depth, and  $g$  the acceleration due to gravity. Partial derivatives are denoted with subscripts (e.g.  $\eta_x \equiv \partial\eta/\partial x$ ).

Following Zakharov (1968) and Craig and Sulem (1993), the FNPF problem can be reformulated as a functional of two free surface variables, i.e., the FSE  $\eta(x, t)$  and the free surface potential  $\tilde{\phi}(x, t) \equiv \phi(x, z = \eta, t)$ . The reformulation allows describing the motion of the bulk fluid by rewriting the nonlinear free surface boundary conditions (2) and (3) as:

$$\eta_t = -\tilde{\phi}_x \eta_x + \tilde{w} \left[ 1 + (\eta_x)^2 \right], \quad (5)$$

$$\tilde{\phi}_t = -g\eta - \frac{1}{2} (\tilde{\phi}_x)^2 + \frac{1}{2} \tilde{w}^2 \left[ 1 + (\eta_x)^2 \right], \quad (6)$$

where  $\tilde{w}(x, t) \equiv \phi_z(x, z = \eta, t)$  denotes the vertical velocity of the water particles on the free surface. Partial derivatives of  $\tilde{\phi}$  are deduced via the chain rule. The two coupled equations (5–6) describe the change rate of  $\tilde{\phi}$  and  $\eta$  in time, with  $\tilde{w}$  being the only unknown variable. Solving  $\tilde{w}$  from the boundary conditions  $(\eta, \tilde{\phi})$  on the free surface is the so-called *Dirichlet-to-Neumann* (DtN) problem. The DtN problem is of fundamental importance for the Zakharov

166 formulation and has been extensively studied. For a more detailed  
167 introduction of various approaches for solving the DtN  
168 problem in different models and the methodology adopted in  
169 W3D, the readers are referred to Tian and Sato (2008); Yates  
170 and Benoit (2015); Raoult et al. (2016), and the references  
171 therein. Here, we first briefly review the key steps of solving  
172 the DtN problem in W3D, then introduce the calculation of  
173 internal kinematics in this model.

First, the physical  $(x, z)$  domain with variable boundaries in the vertical direction  $z \in [-h(x), \eta(x, t)]$  is mapped to a new  $(x, s)$  domain with fixed boundaries  $s \in [-1, 1]$  via the transformation of vertical coordinate:

$$s(x, z, t) = \frac{2z + h^-(x, t)}{h^+(x, t)} \quad (7)$$

174 with  $h^\pm(x, t) \equiv h(x) \pm \eta(x, t)$ . The potential is expressed as  
175  $\varphi(x, s(x, z, t), t) \equiv \phi(x, z, t)$  in the transformed domain, in  
176 which the governing equations are reformulated (not shown  
177 here, see e.g. (Yates and Benoit, 2015)).

Then, the velocity potential is projected onto an orthogonal polynomial basis formed by the set of Chebyshev polynomials of the first kind, denoted  $T_n(s)$ , up to a maximal order denoted  $N_T$ . The potential  $\varphi$  is approximated as:

$$\varphi(x, s, t) \approx \varphi_{N_T}(x, s, t) = \sum_{n=0}^{N_T} a_n(x, t) T_n(s), \quad (8)$$

178 where the coefficients  $a_n(x, t)$ ,  $n = 0, 1, \dots, N_T$ , are the  $N_T + 1$   
179 unknowns of the problem (at each abscissa  $x$ ). The  $N_T$   
180 parameter plays a central role in balancing model accuracy  
181 and computational effort. Benoit et al. (2017) showed the  
182 capability of the linearized W3D model in describing the linear  
183 dispersion, which serves as a good starting point in calibrating  
184 the value of  $N_T$ . Usually, for waves with intermediate  
185 nonlinearity propagating in finite water depth, choosing  
186  $N_T = 7$  yields already very accurate predictions. We may  
187 need to tune  $N_T$  for higher values when considering highly  
188 nonlinear waves (close to breaking) or very deep water  
189 conditions ( $\mu \sim 10$ ). Inserting eq. (8) into the Laplace equation  
190 and adopting the Chebyshev-tau method to remove the dependency  
191 on the  $s$  variable,  $N_T - 1$  linear equations are built.  
192 With two supplementary equations drawn from the free surface  
193 and bottom boundary conditions, the problem is closed  
194 with  $N_T + 1$  linear equations for  $N_T + 1$  unknowns.

Then,  $\tilde{w}$  can be evaluated as:

$$\tilde{w}(x, t) = \frac{2}{h^+(x, t)} \sum_{n=1}^{N_T} a_n(x, t) n^2. \quad (9)$$

195 The DtN problem is thus solved and, with  $\tilde{w}$  known, eqs. (5–  
196 6) can be stepped forward in time. Several time marching  
197 schemes are implemented in W3D. Here we used the explicit  
198 third-order strong stability preserving Runge-Kutta scheme  
199 with a constant time-step  $\Delta t$ . Unless otherwise stated, all  
200 first- and second-order  $x$ -derivatives in the numerical model  
201 are approximated with finite difference (FD) formulas using  
202 a centered stencil of  $N_{sten} = 5$  nodes, which provides fourth-  
203 order formal accuracy in the case of uniform grid size  $\Delta x$ .

## 2.2. Computation of orbital velocities

204 With the  $a_n$  coefficients determined, the orbital velocities  
205 can be evaluated at any point in the fluid domain by taking  
206 the spatial derivatives of  $\phi(x, z, t)$ . To that end, we first  
207 give the expressions of the partial derivatives of the vertical  
208 coordinate  $s(x, z, t)$ , obtained from eq. (7), as:  
209

$$s_x = \frac{h_x^- - s h_x^+}{h^+} = \frac{h_x^-}{h^+} (1 - s) - \frac{\eta_x}{h^+} (1 + s), \quad (10)$$

$$s_z = \frac{2}{h^+}, \quad (11)$$

$$s_t = \frac{h_t^- - s h_t^+}{h^+} = -\frac{\eta_t}{h^+} (1 + s), \quad (12)$$

$$s_{xt} = -\frac{1}{h^+} [\eta_{xt} (1 + s) + \eta_t s_x + s_t h_x^+], \quad (13)$$

$$s_{zt} = -\frac{2\eta_t}{(h^+)^2}. \quad (14)$$

The horizontal velocity  $u(x, z, t) = \phi_x$  and the vertical  
210 velocity  $w(x, z, t) = \phi_z$  are expressed as functions of the  $a_n$   
211 coefficients and their  $x$ -derivatives:  
212

$$u(x, z, t) \approx \frac{\partial \varphi_{N_T}}{\partial x} + s_x \frac{\partial \varphi_{N_T}}{\partial s} \quad (15)$$

$$= \sum_{n=0}^{N_T} a_{n,x} T_n + \frac{h_x^- - s h_x^+}{h^+} \sum_{n=1}^{N_T} a_n T_{n,s},$$

$$w(x, z, t) \approx s_z \frac{\partial \varphi_{N_T}}{\partial s} = \frac{2}{h^+} \sum_{n=1}^{N_T} a_n T_{n,s}. \quad (16)$$

213 These expressions can be projected on the basis formed  
214 by the orthogonal polynomials  $\{T_n, n = 0, \dots, N_T\}$  using the  
215 inner product defined in Eq. (24) of Raoult et al. (2019), and  
216 equivalently reformulated in a compact form, as:

$$u(x, z, t) = \sum_{p=0}^{N_T} a_p^u T_p, \quad (17)$$

$$w(x, z, t) = \sum_{p=0}^{N_T-1} a_p^w T_p, \quad (18)$$

with the coefficients  $a_p^u$ ,  $p = 0, \dots, N_T$ , and  $a_p^w$ ,  $p = 0, \dots, N_T - 1$ ,  
217 given analytically by:  
218

$$a_p^u = a_{p,x} + \frac{1}{h^+} (h_x^- S_{1p} - h_x^+ S_{2p}), \quad (19)$$

$$a_p^w = \frac{2}{h^+} S_{1p}. \quad (20)$$

where

$$S_{1p} \equiv \sum_{n=1}^{N_T} a_n B_{p01n}, \quad (21)$$

$$S_{2p} \equiv \sum_{n=1}^{N_T} a_n B_{p11n}. \quad (22)$$

220 The expressions of the terms  $B_{p01n}$  and  $B_{p11n}$  are already  
 221 defined in appendix B of Raoult et al. (2019), thus not re-  
 222 peated here for brevity. We note the maximum degree of  $T_n$   
 223 polynomials is  $N_T$  for  $u$  in eq. (17) and  $N_T - 1$  for  $w$  in  
 224 eq. (18).

### 2.3. Computation of Eulerian accelerations

225 The local (Eulerian) accelerations  $ax(x, z, t) = u_t$  and  
 226  $az(x, z, t) = w_t$  are obtained as:

$$ax \approx \sum_{n=0}^{N_T} a_{n,xt} T_n + \sum_{n=1}^{N_T} [s_t a_{n,x} + s_{xt} a_n + s_x a_{n,t}] T_{n,s} \quad (23)$$

$$+ s_x s_t \sum_{n=2}^{N_T} a_n T_{n,ss},$$

$$az \approx \sum_{n=1}^{N_T} (s_z a_{n,t} + s_{zt} a_n) T_{n,s} + s_z s_t \sum_{n=2}^{N_T} a_n T_{n,ss} \quad (24)$$

228 As for the velocity components, these expressions can be  
 229 projected on the basis  $\{T_n, n = 0, \dots, N_T\}$  and equivalently  
 230 reformulated as:

$$ax(x, z, t) = \sum_{p=0}^{N_T} a_p^{ax} T_p, \quad (25)$$

$$az(x, z, t) = \sum_{p=0}^{N_T-1} a_p^{az} T_p, \quad (26)$$

231 with the coefficients  $a_p^{ax}, p = 0, \dots, N_T$ , and  $a_p^{az}, p = 0, \dots, N_T -$   
 232 1, given analytically by:

$$a_p^{ax} = a_{p,xt} - \frac{2}{h^+} \left[ \eta_t (S_{3p} + S_{4p}) + (\eta_{xt} - 2 \frac{\eta_t \eta_x}{h^+}) \times \right. \quad (27)$$

$$\left. (S_{1p} + S_{2p}) - 2 \frac{\eta_t h_x}{h^+} S_{2p} - h_x^- S_{5p} + h_x^+ S_{6p} \right]$$

$$+ \frac{\eta_t}{(h^+)^2} [h_x (S_{9p} - S_{7p}) + \eta_x (S_{7p} + 2S_{8p} + S_{9p})],$$

$$a_p^{az} = \frac{2}{h^+} (S_{5p} - \frac{\eta_t}{h^+} (S_{1p} + S_{7p} + S_{8p})), \quad (28)$$

233 where

$$S_{3p} \equiv \sum_{n=1}^{N_T} a_{n,x} B_{p01n} = \frac{\partial S_{1p}}{\partial x}, \quad (29)$$

$$S_{4p} \equiv \sum_{n=1}^{N_T} a_{n,x} B_{p11n} = \frac{\partial S_{2p}}{\partial x}, \quad (30)$$

$$S_{5p} \equiv \sum_{n=1}^{N_T} a_{n,t} B_{p01n} = \frac{\partial S_{1p}}{\partial t}, \quad (31)$$

$$S_{6p} \equiv \sum_{n=1}^{N_T} a_{n,t} B_{p11n} = \frac{\partial S_{2p}}{\partial t}, \quad (32)$$

$$S_{7p} \equiv \sum_{n=1}^{N_T} a_n B_{p02n}, \quad (33)$$

$$S_{8p} \equiv \sum_{n=1}^{N_T} a_n B_{p12n}, \quad (34)$$

$$S_{9p} \equiv \sum_{n=1}^{N_T} a_{n,t} B_{p22n}. \quad (35)$$

234 The expressions of the terms  $B_{p02n}$ ,  $B_{p12n}$  and  $B_{p22n}$  are also  
 235 provided in Raoult et al. (2019).

236 To compute the accelerations, we need to provide the  
 237 time derivatives of the FSE, denoted  $\eta_t$  (appearing in eqs. (12)–  
 238 (14)), and those of the  $a_n$  coefficients, denoted  $a_{n,t}$  (appear-  
 239 ing in eqs. (23)–(24)). For  $\eta_t$ , we use the expression at the  
 240 right-hand side of eq. (5). Then,  $\eta_{xt}$  can be obtained from  $\eta_t$   
 241 with the 5-node centered FD scheme mentioned above.

242 The  $a_{n,t}$  terms are computed using a high-order FD scheme  
 243 in time, with two options available:

- calculation of kinematics after completion of the sim- 244  
 245 ulation: a centered scheme around the current time  
 246 instant is used, with again a 5-node stencil:

$$a_{n,t}(t) = \frac{1}{12\Delta t} [8 (a_n(t + \Delta t) - a_n(t - \Delta t)) - (a_n(t + 2\Delta t) - a_n(t - 2\Delta t))] \quad (36)$$

- calculation in the course of the simulation: an upwind 247  
 248 scheme is used, with the current value of  $a_n$  and the  
 249 three previous ones (giving third-order formal accu-  
 250 racy):

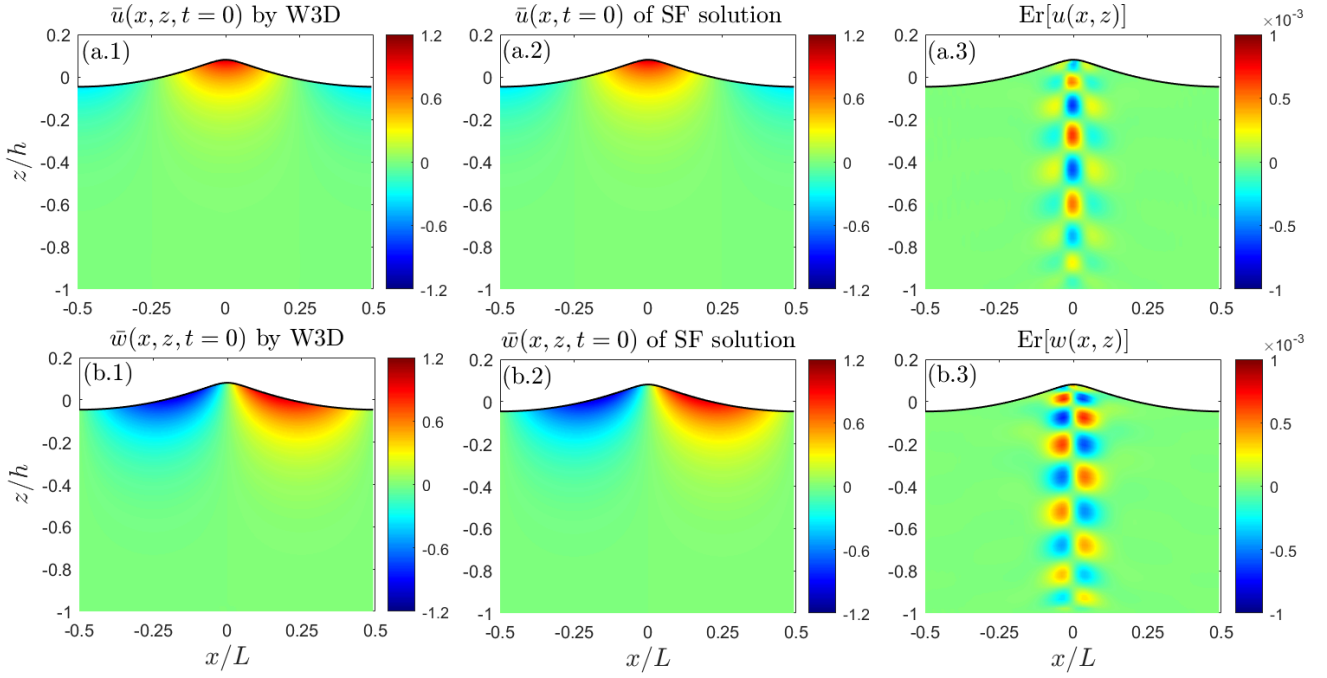
$$a_{n,t}(t) = \frac{1}{\Delta t} \left[ \frac{11}{6} a_n(t) - 3a_n(t - \Delta t) + \frac{3}{2} a_n(t - 2\Delta t) - \frac{1}{3} a_n(t - 3\Delta t) \right]. \quad (37)$$

251 As for  $\eta_{xt}$ , the terms  $a_{n,xt}$  are then computed from the terms  
 252  $a_{n,t}$  using again a centered 5-node FD scheme in space.

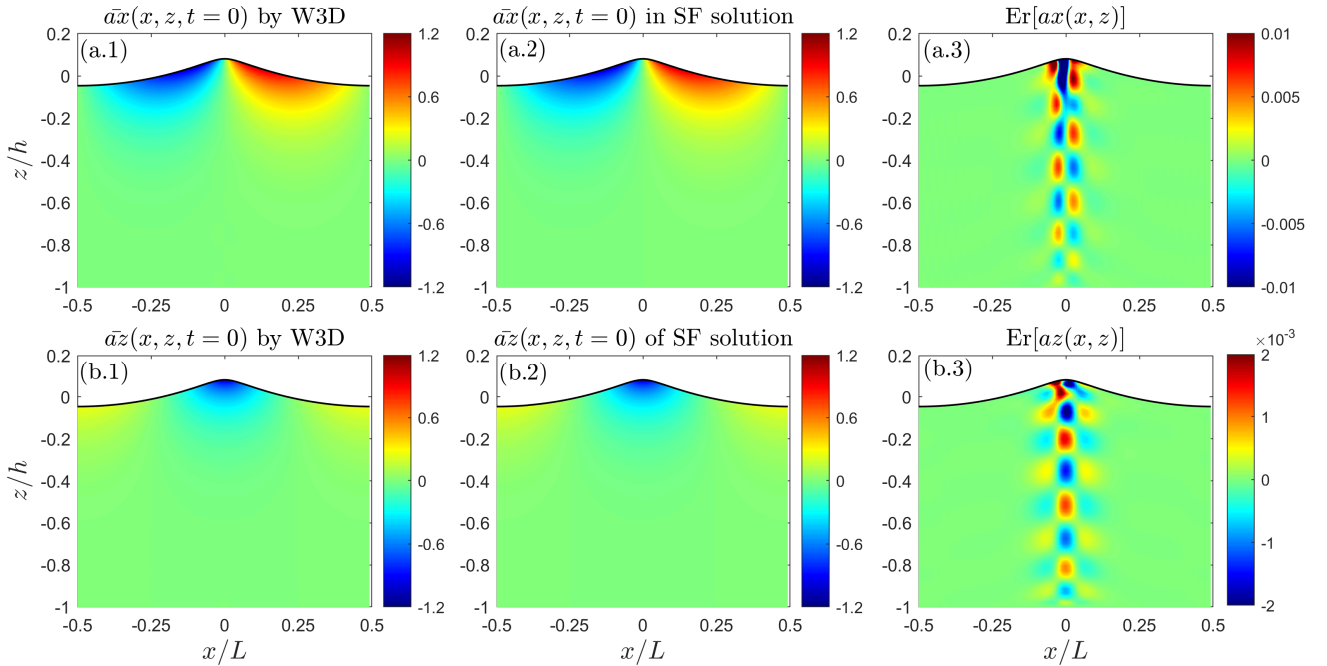
## 3. Verification for regular nonlinear waves in uniform water depth

253 In this section, we demonstrate the accuracy of the pro-  
 254 posed formulas for computing wave kinematics of a highly  
 255 nonlinear periodic wave propagating with permanent form in  
 256 uniform water depth. For a quantitative assessment, the nu-  
 257 merical solution of SF theory is considered, which provides  
 258 a solution of arbitrarily high accuracy for this case. The SF  
 259 solution is obtained here by imposing that the mean Eulerian  
 260 flow velocity at any point below the wave trough is null.  
 261

262 Given the still water depth  $h$ , wavelength  $L$  (or, equiva-  
 263 lently, wave number  $k = 2\pi/L$ ), and wave height  $H$ , a reg-  
 264 ular wave condition is defined. Here, we choose a challeng-  
 265 ing case: (i) the relative water depth is set to  $h/L = 1$  (i.e.  
 266  $kh = 2\pi$ ), achieving twice the traditionally accepted ‘deep  
 267 water’ threshold ( $kh = \pi$ ); (ii) a very high value of the wave  
 268



**Figure 1:** Normalized velocity fields ( $\bar{u}, \bar{w}$ ) at  $t = 0$  predicted by the W3D model in panels (a.1) and (b.1) and the corresponding SF solution in panels (a.2) and (b.2), respectively, and the relative error of the velocities obtained with two methods in panels (a.3) and (b.3). In each panel, the wave profile is outlined with a thick black line on the free surface.



**Figure 2:** Normalized acceleration fields ( $\bar{a}_x, \bar{a}_z$ ) at  $t = 0$  predicted by the W3D model in panels (a.1) and (b.1) and the corresponding SF solution in panels (a.2) and (b.2), respectively, and the relative error of the accelerations obtained with two methods in panels (a.3) and (b.3). In each panel, the wave profile is outlined with a thick black line on the free surface.

269 steepness is selected  $H/L = 12.73\%$  (i.e.  $kH/2 = 0.40$ ).  
 270 According to the approximate relationship giving the maximum  
 271 stable wave height, expressed as eq. (32) of Fenton  
 272 (1990), such a wave steepness represents 90% of the maximum  
 273 wave height for the chosen relative water depth. In

practice, the wavelength is set as  $L = 64$  m. We use 25  
 Fourier coefficients for the expansion of the SF, providing a  
 converged solution. 274 275 276

The spatial profiles of FSE ( $\eta$ ) and free surface potential  
 ( $\tilde{\phi}$ ) computed with the SF method are given as initial

conditions to the W3D simulation. We run the W3D simulation over a periodic domain covering one wavelength exactly, which is discretized with 128 cells of constant size  $\Delta x = L/128 = 0.5$  m. The theoretical wave period obtained from the SF solution is  $T_{\text{SF}} \equiv 2\pi/\omega_{\text{SF}} \approx 5.916$  s. We note that, as expected, this period is shorter than the period obtained from the dispersion relation of linear waves:

$$T_{\text{lin}} = \frac{2\pi}{\omega_{\text{lin}}} = \frac{2\pi}{\sqrt{gk \tanh(kh)}} \approx 6.402 \text{ s.} \quad (38)$$

We note  $T_{\text{SF}}$  is smaller than  $T_{\text{lin}}$  because the SF wave solution follows a nonlinear dispersion relation which predicts a faster wave phase speed for a wave with the same wavelength in comparison to the prediction of linear dispersion. In the simulation, a constant time step  $\Delta t = T_{\text{SF}}/128 \approx 0.046$  s is chosen, which results in a Courant–Friedrichs–Lewy number  $\text{CFL} \equiv (L\Delta t)/(T_{\text{SF}}\Delta x) = 1$ . Initial conditions for  $(\eta(x), \tilde{\phi}(x))$  at  $t = -3\Delta t$  from the SF solution are provided as input to W3D, and time integration is performed over  $3\Delta t$  to reach  $t = 0$  s where results are compared with the SF solution. As  $kh$  is quite large here and waves are strongly nonlinear, we choose  $N_T = 11$  in the W3D simulation after calibration.

The computed velocities and accelerations are presented hereafter in a dimensionless way, normalized with the modulus of the corresponding quantities expressed from linear wave theory (under the same condition  $(h, L, H)$  as for the SF solution) at mean water level (i.e.,  $z = 0$ ), that is  $\bar{u} \equiv u/U_{\text{lin}}$ ,  $\bar{w} \equiv w/W_{\text{lin}}$ ,  $\bar{ax} \equiv ax/Ax_{\text{lin}}$ ,  $\bar{az} \equiv az/Az_{\text{lin}}$ , with:

$$U_{\text{lin}} = gka_{\text{lin}}/\omega_{\text{lin}} \quad (39)$$

$$W_{\text{lin}} = a_{\text{lin}}\omega_{\text{lin}} \quad (40)$$

$$Ax_{\text{lin}} = gka_{\text{lin}} \quad (41)$$

$$Az_{\text{lin}} = a_{\text{lin}}\omega_{\text{lin}}^2 \quad (42)$$

with  $a_{\text{lin}} \equiv H/2$ . The relative errors are made dimensionless by the maximum value of the modulus of the SF solution, e.g. for the vertical velocity component:

$$\text{Er}[w(x, z)] = \frac{w_{\text{W3D}}(x, z) - w_{\text{SF}}(x, z)}{|w_{\text{SF}}(x, z = \eta)|_{\text{max}}}. \quad (43)$$

The non-dimensional velocity fields ( $\bar{u}$  and  $\bar{w}$ ) at  $t = 0$  (with wave crest at  $x = 0$ ) in both W3D simulation and SF theory are displayed in Fig. 1, with their relative errors provided. As shown in Fig. 1(a.1–a.2) and (b.1–b.2), the velocity fields ( $\bar{u}$ ,  $\bar{w}$ ) are well predicted by the W3D model, with the coloured maps of theoretical and simulated velocity components being visually indistinguishable from each other. The plots (a.3–b.3) of relative error show the differences appear mainly below the wave crest and remain less than 0.1% all over the water column. The velocity field underneath the wave trough is well predicted, with the maximum relative error below  $10^{-4}$ .

The acceleration at  $t = 0$  is also computed, which involves the backward FD scheme (37) to evaluate the time derivative of the coefficients  $a_n(x, t = 0)$ . Fig. 2 shows the

non-dimensional acceleration fields ( $\bar{ax}$  and  $\bar{az}$ ) at  $t = 0$  in both W3D simulation and SF theory, as well as the relative error between them. Again, the simulated and theoretical acceleration  $(x, z)$ -maps are visually identical in the current colour scale of panels (a.1–a.2) and (b.1–b.2) in Fig. 2. The relative error of  $ax$  in panel (a.3) achieves the most prominent values beneath the wave crest. It is larger than the ones reached for the velocity components but remains bounded by 1%. The relative error is lower for the vertical acceleration  $az$  in panel (b.3), not exceeding 0.5%. These low error levels confirm the accuracy of the scheme chosen to evaluate  $a_{n,t}$ , bearing in mind that the model has evolved the input initial solution for 3 time-steps to reach the state shown in Fig. 2.

In conclusion, the W3D schemes can model with accuracy the wave kinematics beneath highly nonlinear (close-to-breaking) regular waves, without any sign of singularity. Larger errors are observed below the wave crest (never exceeding 0.1% for the velocity components and 1% for the acceleration components in the case shown here). It is also observed that the relative errors of the horizontal components of velocity and acceleration are a bit higher than those of their vertical counterparts.

## 4. Experimental validation for regular nonlinear waves in variable water depth

### 4.1. Experimental configuration

In this section, W3D's performance is evaluated by simulating an experimental test for which detailed measurements of the FSE and orbital velocity beneath regular waves are available. We chose a case with regular nonlinear waves propagating over an inhomogeneous medium performed at the hydrodynamics laboratory of the Department of Mathematics of the University of Oslo (Norway) and introduced in Lawrence et al. (2021a). The wave flume is 24.6 m long and 0.5 m wide. A trapezoidal-shaped bar was installed on the bottom, consisting of plane ascending and descending slopes with 1/3.81 gradient and 1.6 m length each, and a plateau of 1.6 m length that connects the two slopes. The water depth is changed from  $h_1 = 0.53$  m before and after the bar to  $h_2 = 0.11$  m atop the bar, so that the height of the bar above the horizontal seabed is 0.42 m (see Fig. 3). A piston-type wavemaker is located at one end of the flume, and a wave absorbing zone at the other. Given the origin of the  $x$ -axis set at the beginning of the bar crest, the abscissa of the wavemaker is  $-12.36$  m.

The measuring devices contain four ultrasonic wave probes to record the FSE, and one Nortek "Vectrino" acoustic Doppler velocimeter (ADV) to record the velocity at an elevation  $z_0 = -0.05$  m below the SWL, giving  $z_0 \approx -0.1h_1 \approx -0.45h_2$ . The test with regular waves presented in Section 3.4 of Lawrence et al. (2021a) is considered here. In this test, the wave frequency is set as  $f_0 = 0.7$  Hz (i.e.,  $T_0 \approx 1.43$  s, the corresponding wavelength before the bar is  $L_1 = 2\pi/k_1 = 2.69$  m according to the linear wave dispersion relation) and the wave amplitude  $H_0 = 2a_0 = 0.0270$  m. In such a configuration, the incident wave steepness is  $\epsilon_1 \equiv k_1a_0 =$

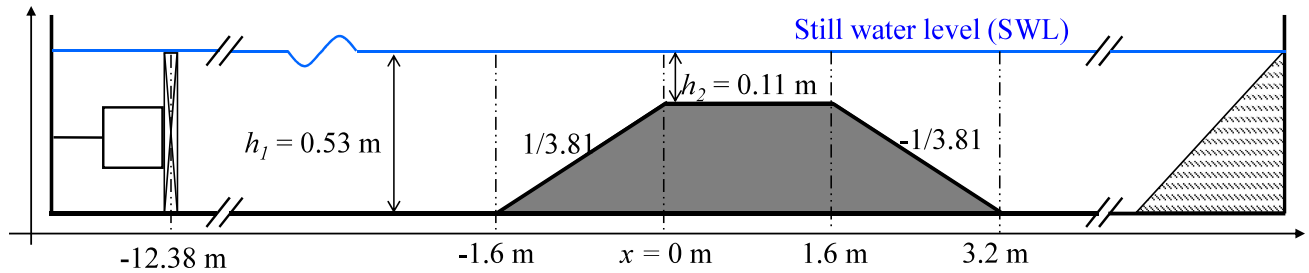


Figure 3: Sketch of the bathymetry used in the experiments reported in Lawrence et al. (2021a)

366 0.032 (or  $H_0/h_1 = 0.051$ ) and the relative water depth  $\mu_1 \equiv$   
 367  $k_1 h_1 = 1.237$  (or  $h_1/L_1 = 0.197$ ), resulting in an Ursell  
 368 number  $Ur_1 \equiv (H_0/h_1) (L_1/h_1)^2 = 13.12$ . Over the bar,  
 369 the parameters are then  $\epsilon_2 = 0.059$  (or  $H_0/h_2 = 0.246$ ),  
 370  $\mu_2 = 0.483$  (or  $h_2/L_2 = 0.077$ ) and  $Ur_2 = 414.93 >$   
 371  $100$ . Therefore the relative importance of nonlinearity is  
 372 considerably enhanced by the shoal. The wave field near the  
 373 wavemaker is of intermediate nonlinearity, yet it results in  
 374 highly nonlinear but non-breaking waves over the bar. The  
 375 same configuration was tested 31 times, such that the FSE  
 376 was measured at 124 positions and the horizontal velocity at  
 377 31 positions. The measurements were performed in a suffi-  
 378 ciently short time to avoid the effects of reflection at the end  
 379 of the flume.

## 380 4.2. Numerical model setup

381 In the numerical flume, the waves are generated and damped  
 382 within two relaxation zones of 8 m in length (approximately  
 383  $3L_1$ ) located at both ends of the numerical flume. The effec-  
 384 tive computation domain for wave propagation (i.e., exclud-  
 385 ing the two relaxation areas) starts at  $x = -5.5$  m and ends  
 386 at  $x = 10.3$  m. The governing equations are discretized with  
 387 constant space and time step,  $\Delta x = 0.04$  m and  $\Delta t = 0.02$  s,  
 388 respectively. Such a choice results in CFL= 0.94 in the  
 389 deeper flat regions and CFL= 0.50 over the bar crest. The  
 390 polynomial order is set to  $N_T = 7$ .

## 391 4.3. Free surface elevation

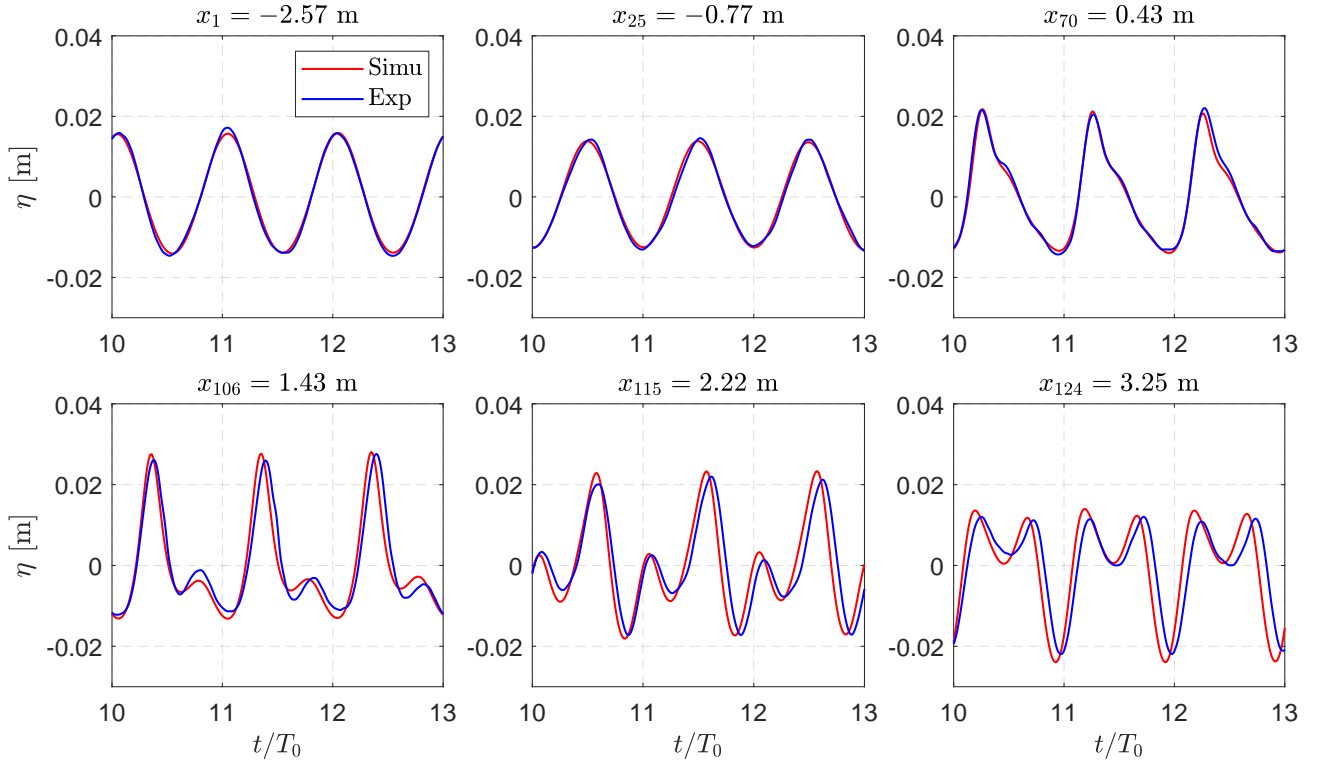
392 Comparisons of the measured and computed FSE time  
 393 profiles are shown in Fig. 4 at 6 positions over a time win-  
 394 drow of  $3T_0$ . Before the bar (probes #1 and #25), the waves  
 395 are nearly symmetric in both the horizontal and vertical di-  
 396 rections. As waves propagate over the bar (probes #70 and  
 397 #106), the wave profiles become asymmetric, and secondary  
 398 crests manifest. These secondary crests are related to the  
 399 development of high-order harmonics that propagate with  
 400 a different velocity in comparison to the carrier wave. As  
 401 waves propagate over the de-shoaling slope (probes #115  
 402 and #124), due to the presence of bound and free super-  
 403 harmonics (propagating with different velocities) the free sur-  
 404 face time profile is very variable from one position to an-  
 405 other. It should be noticed that after the de-shoaling slope  
 406 (probe #124), the wave profile is asymmetric, with sharper  
 407 troughs and flatter crests. The evolution of the wave pro-  
 408 file along the flume is very well reproduced by the model,

409 despite some slight phase shifts at probes #115 and #124.

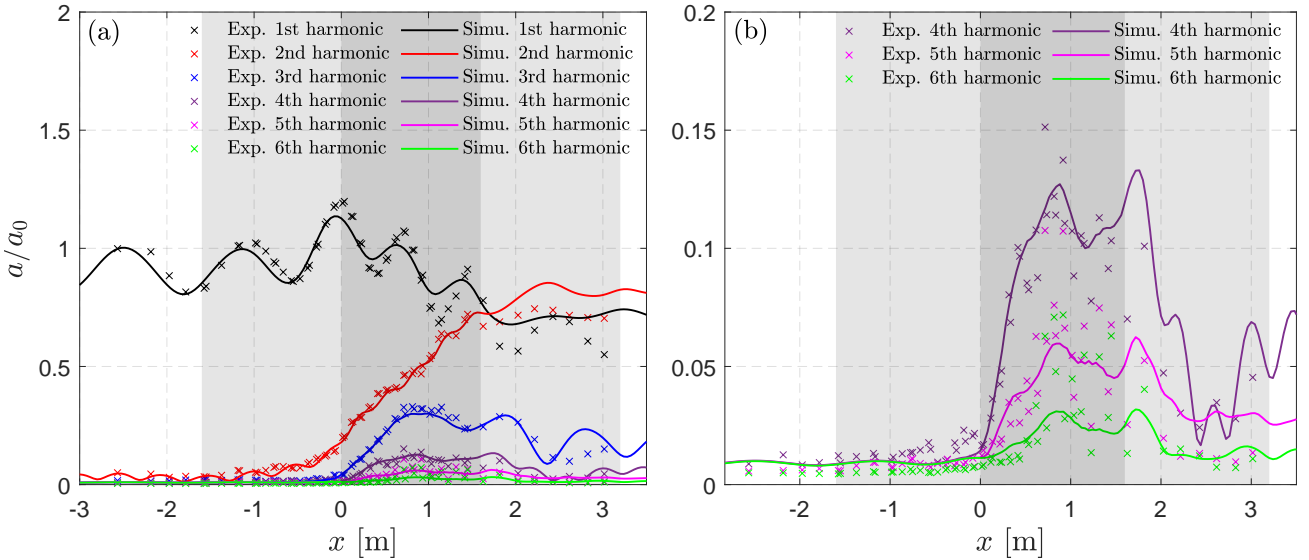
410 To better illustrate the evolution of the waves over the  
 411 varying bathymetry and to assess the magnitude of nonlinear  
 412 effects, a Fourier analysis is applied to both the measured and  
 413 simulated time series. Fig. 5(a) shows the spatial evolution  
 414 of the amplitudes of the first six harmonics (i.e. wave com-  
 415 ponents with frequencies  $nf_0$ , with  $n = 1, \dots, 6$ ), normal-  
 416 ized by the amplitude measured at probe #1, denoted as  $a_0$ . The  
 417 fourth to sixth harmonics are duplicated in Fig. 5(b) with a  
 418 reduced range in y-axis to have a better view. The evolution  
 419 of the primary component shows some oscillations before  
 420 and over the bar, this is expected to be the result of reflection  
 421 (by both the ascending and descending slopes). Over the bar  
 422 crest, the amplitude of the primary component starts to de-  
 423 crease due to the enhancement of the amplitudes of the high-  
 424 order super-harmonics. The second harmonic is consider-  
 425 ably increased over the bar crest and remains on a high level.  
 426 Eventually, the amplitude of this second harmonic becomes  
 427 comparable to the primary component over the de-shoaling  
 428 slope. The third harmonic is also increased over the bar  
 429 with some oscillations, and it decreases over the de-shoaling  
 430 slope. The evolution trends of the first three harmonics are  
 431 very closely reproduced by the model. In Fig. 5(b), the evo-  
 432 lution trend of the fourth to sixth harmonics is similar to that  
 433 of the third harmonics and is also well captured by the model.  
 434 It is seen that measured results scatter around the simulated  
 435 results, this could be explained by the fact that the fourth to  
 436 sixth harmonics are of small magnitudes and may be influ-  
 437 enced by noise during the measurements.

438 As shown in Fig. 4, waves become asymmetric in both  
 439 vertical and horizontal directions when propagating over the  
 440 submerged bar. The magnitudes of the vertical and hori-  
 441 zontal asymmetry are measured by the skewness of the FSE  
 442 and its Hilbert transform (Elgar and Guza, 1985),  $\lambda_3(\eta)$  and  
 443  $\lambda_3(\mathcal{H}(\eta))$  (known as asymmetry parameter, with  $\mathcal{H}$  denoting  
 444 the Hilbert transform operator), respectively. The kurtosis is  
 445 a measure of the extreme values in a time series: for irregu-  
 446 lar waves, it is related to the occurrence probability of freak  
 447 waves, and for regular waves, it is still an important index  
 448 that characterizes wave nonlinearity. For a normalized ran-  
 449 dom variable  $\bar{X}$  with a zero mean and a unit variance, the  
 450 skewness, asymmetry and kurtosis parameters are defined  
 451 respectively as:

$$\lambda_3(\bar{X}) = \langle \bar{X}^3 \rangle, \quad (44)$$



**Figure 4:** Comparison of measured and computed FSE temporal profiles at 6 locations for the nonlinear regular wave experiment of Lawrence et al. (2021a)



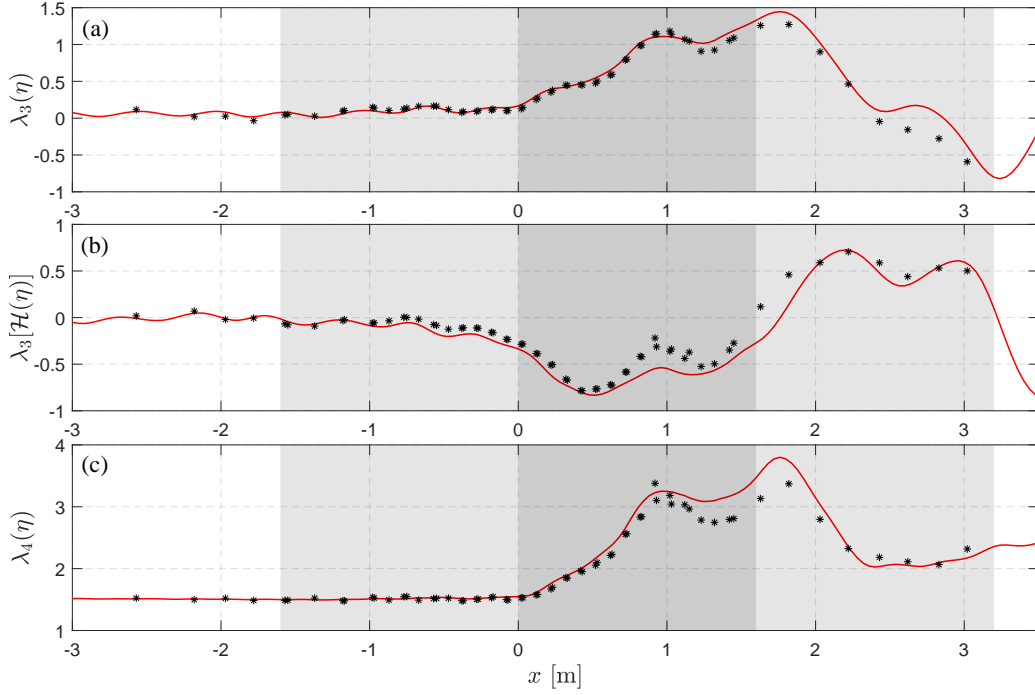
**Figure 5:** (a) Spatial evolution of measured and computed normalized amplitudes of the first 6 harmonics of the FSE for the nonlinear regular wave experiment of Lawrence et al. (2021a). (b) Close-up view of the fourth to sixth harmonics, with a reduced extent of the vertical axis. The gray areas indicate the extent of the submerged bar.

$$\lambda_3[H(\bar{X})] = \langle H(\bar{X})^3 \rangle, \quad (45)$$

$$\lambda_4(\bar{X}) = \langle \bar{X}^4 \rangle. \quad (46)$$

452 where  $\langle \cdot \rangle$  denotes a mean operator,  $\bar{X}$  could be normalized  
 453 FSE, velocity or acceleration. In this subsection, we take  $\bar{X}$   
 454 as  $\bar{\eta}$ . As we are discussing regular waves, in practice, the

455 skewness, asymmetry and kurtosis are computed for the averaged  
 456 free surface profile over one wave period. In the linear framework,  
 457 the skewness and asymmetry of a sinusoidal wave over one (or multiple)  
 458 period(s) are 0, while the kurtosis is expected to be 1.5. The deviation  
 459 of these parameters from their linear expectation is an indication of the magni-  
 460



**Figure 6:** Spatial evolution of statistical moments of the measured and computed FSE for the nonlinear regular wave experiment of Lawrence et al. (2021a). The gray areas indicate the extent of the submerged bar.

461 tude of nonlinearity.

462 The evolution of skewness, asymmetry and kurtosis is  
 463 shown in Fig. 6. It is observed that in Fig. 6(a), the skewness  
 464 is increased over the bar crest, indicating waves with sharper  
 465 crests and flatter troughs. It is then significantly decreased  
 466 over the de-shoaling area, indicating that deep troughs and  
 467 smaller crests develop in the wave profile in this area. In  
 468 Fig. 6(b), the evolution of the asymmetry parameter indicates  
 469 that the wave profile first leans forward then backward,  
 470 and eventually restores a symmetric shape in the horizontal  
 471 direction. Symmetrical wave profiles in both horizontal and  
 472 vertical directions are not recovered after the de-shoaling  
 473 zone, at least within the current spatial scale. In Fig. 6(c),  
 474 the kurtosis of FSE shows a similar evolution trend as the skewness,  
 475 it starts to deviate from the linear expectation 1.5 after  
 476 the up-slope and increases significantly over the bar crest.  
 477 Eventually, the kurtosis does not recover 1.5 after passing  
 478 over the down-slope, and remains higher than this linear  
 479 expectation instead.

#### 480 4.4. Orbital wave velocities at $z_0 = -0.05$ m

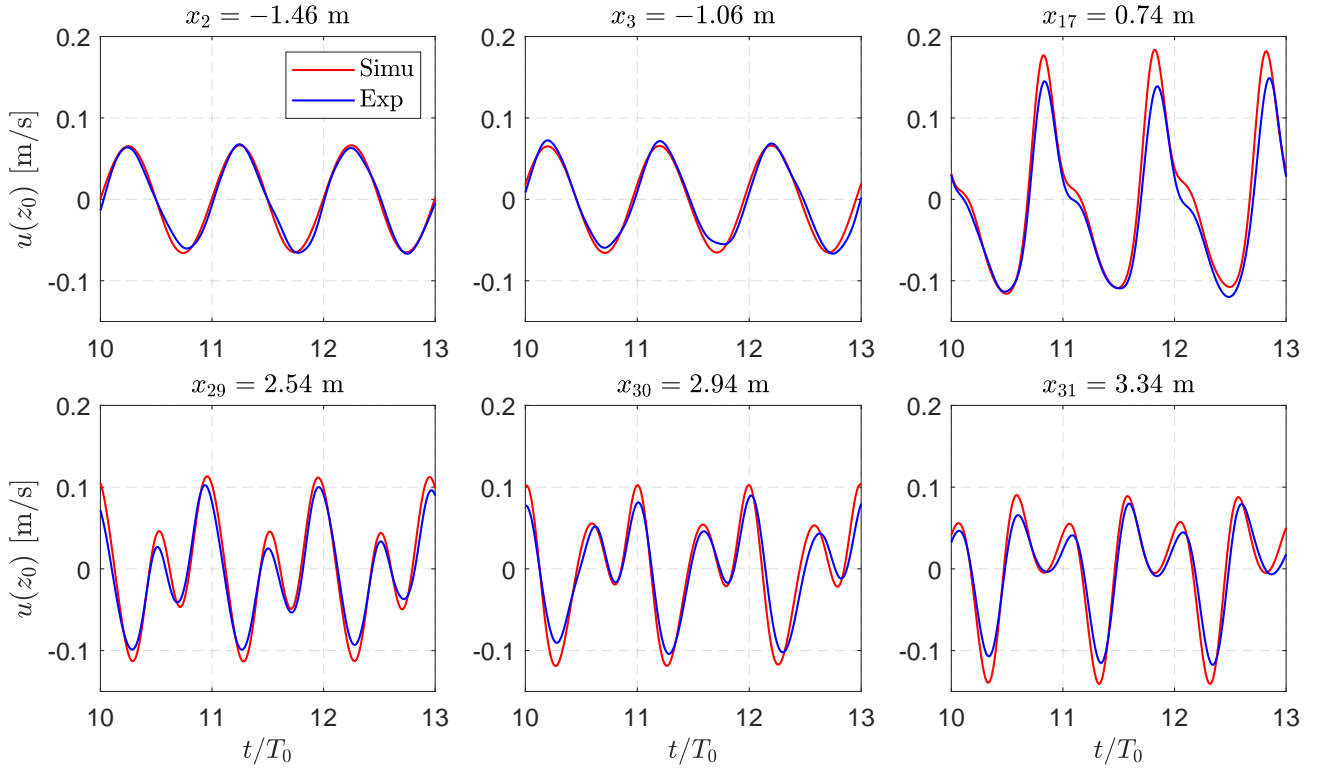
481 Fig. 7 shows the evolution profiles of  $u(z_0)$  ( $z_0 = -0.05$  m)  
 482 at the same six positions as in Fig. 4, within again a time  
 483 frame of  $3T_0$ . In general, the evolution of  $u(z_0)$  profile is  
 484 quite similar to that of  $\eta$ . The main difference between the  
 485 evolution of  $u(z_0)$  and  $\eta$  is that the contribution of the high-  
 486 order harmonics is enhanced in the time series of  $u(z_0)$ . The  
 487 enhancement is proportional to the wave number of high-  
 488 order harmonics, and this explains the more pronounced sec-  
 489 ondary peaks over and after the submerged bar (at locations  
 490 of probes #29 and #30 for instance).

491 The harmonic analysis is performed for both the mea-  
 492 sured and computed horizontal velocity  $u(z_0)$ . It is shown  
 493 in Fig. 8(a) that the magnitudes and the evolution trends of  
 494  $u(z_0)$  are reproduced by the model with high accuracy to the  
 495 third order. In Fig. 8(b), the measured fourth to sixth-order  
 496 harmonics of  $u(z_0)$  are slightly lower than the model predic-  
 497 tions. Again, the measured fourth to sixth harmonics show  
 498 some oscillations around their mean levels, which could be  
 499 related to digital noise. The spatial evolution of skewness,  
 500 asymmetry and kurtosis is displayed in Fig. 9. The evolution  
 501 trends of these statistical parameters of  $u(z_0)$ , in particular  
 502 the skewness and asymmetry parameters, are very similar to  
 503 those of  $\eta$ , indicating that the shapes of the horizontal ve-  
 504 locity and FSE profiles evolve similarly.

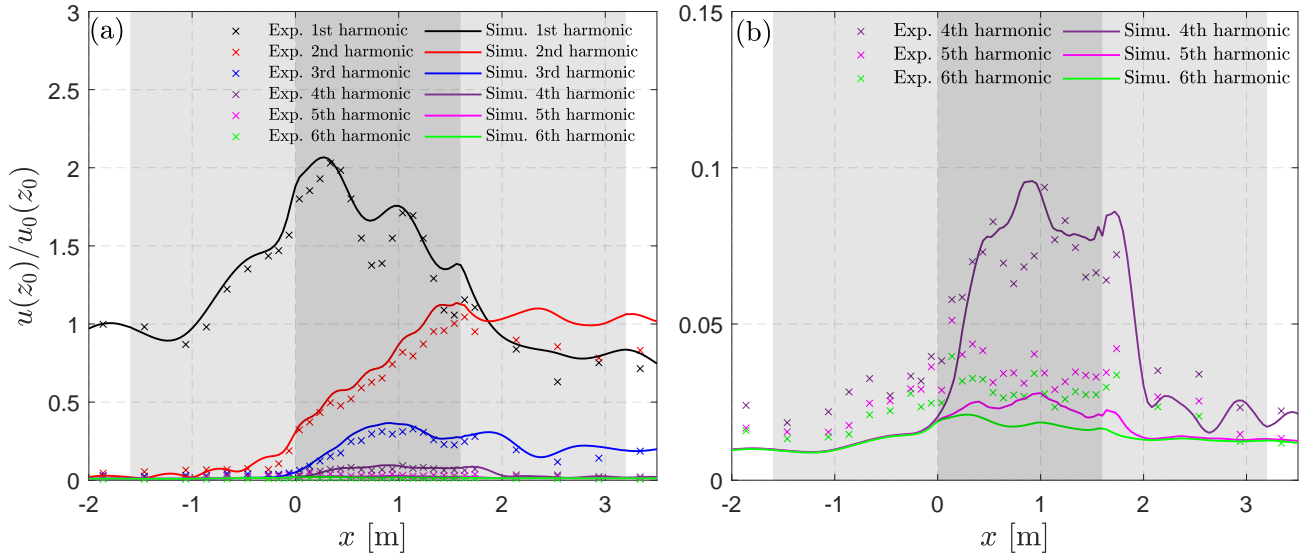
505 To sum up, the various comparisons confirm the high  
 506 accuracy of the W3D model in computing the horizontal ve-  
 507 locity beneath nonlinear waves. In contrast, Lawrence et al.  
 508 (2021a) indicate that they had to use two different models to  
 509 simulate this case, namely a HOS-type model to predict the  
 510 FSE field and a variational Boussinesq model to calculate  
 511 the velocities in the fluid domain.

#### 512 4.5. Eulerian wave accelerations at $z_0 = -0.05$ m

513 The horizontal acceleration  $ax(z_0) = (\partial u / \partial t)|_{z=z_0}$  is not  
 514 directly available from the measurements. Here, it is esti-  
 515 mated by computing the time derivative of the measured  
 516  $u(z_0)$  signal, using a centered FD scheme over a stencil of 5  
 517 signal points (similar to eq. (36)). In the model, the accel-  
 518 eration is directly evaluated using eq. (25). For comparison,  
 519 we also computed it by deriving the simulated  $u(z_0)$  time se-  
 520 ries with respect to time, using the same FD scheme as for



**Figure 7:** Comparison of measured and computed horizontal velocity  $u(z_0)$  temporal profiles at 6 locations for the nonlinear regular wave experiment of Lawrence et al. (2021a).

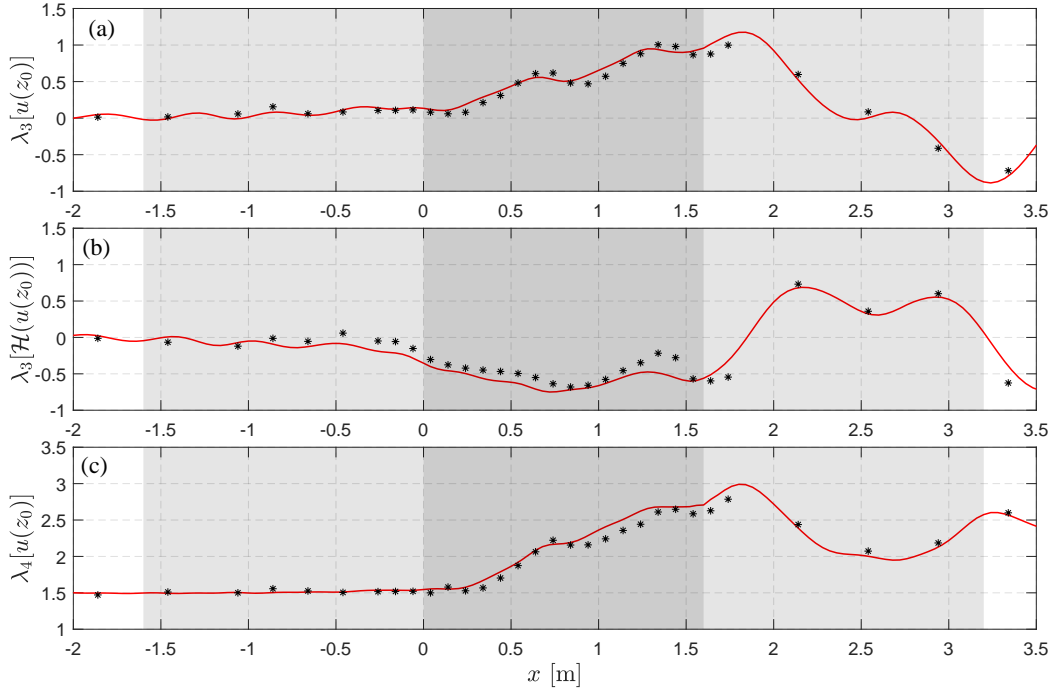


**Figure 8:** (a) Spatial evolution of normalized measured and computed amplitudes of the first 6 harmonics of the horizontal velocity  $u(z_0)$  for the nonlinear regular wave experiment of Lawrence et al. (2021a). (b) Close-up view of the fourth to sixth harmonics, with a reduced extent of the vertical axis. The gray areas indicate the extent of the submerged bar.

521 the measurements.

522 In Fig. 10, the time profiles of  $ax(z_0)$  at 6 locations (same  
523 as in Fig. 7) are shown. In each panel of Fig. 10, the time  
524 profiles of  $ax(z_0)$  derived from the measured and simulated  
525  $u(z_0)$  with FD method, and simulated  $ax(z_0)$  evaluated with  
526 eq. (25) are superimposed. It is seen that the  $ax(z_0)$  directly

527 computed with eq. (25) is in good agreement with that calcu-  
528 lated from  $u(z_0)$  with the FD method, thus validating our im-  
529 plementation of the acceleration computation. In the follow-  
530 ing, unless otherwise stated, simulated  $ax(z_0)$  refers to the  
531 results obtained with eq. (25); measured  $ax(z_0)$  refers to the  
532 acceleration evaluated from the measured  $u(z_0)$  with the FD



**Figure 9:** Spatial evolution of statistical moments of the measured and computed horizontal velocity  $u(z_0)$  for the nonlinear regular wave experiment of Lawrence et al. (2021a). The gray areas indicate the extent of the submerged bar.

533 method. The agreement between the simulated and the measured  $ax(z_0)$  is fairly good throughout the domain, although  
 534 some differences in the magnitude of crests and troughs develop after the up-slope.  
 535

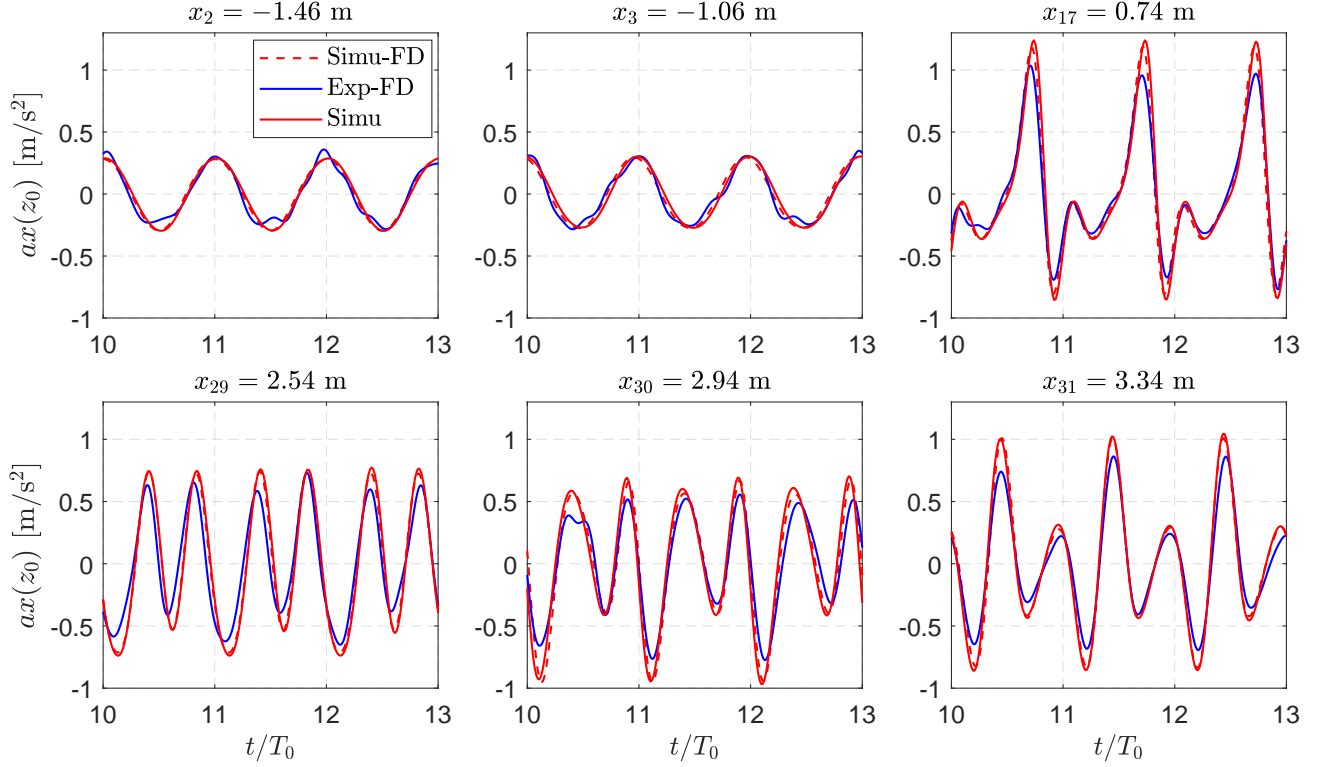
536  
 537 The harmonic analysis is performed for both the simulated and measured  $ax(z_0)$  time series, with the evolution of  
 538 the amplitudes of the first six harmonics displayed in Fig. 11. In Fig. 11(a), it is seen that the agreement between the simulated and measured results is reasonable. Yet, the second-  
 539 order harmonic is overestimated in the simulation starting around  $x = 0.5$  m, leading to higher crests and deeper troughs in the simulated  $ax(z_0)$  time series. It is noted that the amplitudes of the super-harmonics (second and higher orders)  
 540 are increasing when comparing the spectral evolution of  $\eta$ ,  $u(z_0)$ , and  $ax(z_0)$  in space. This is the result of the free super-harmonics excited by the shoal, which produce larger  
 541 amplitudes of velocity and acceleration in comparison to the bound components.  
 542  
 543  
 544  
 545  
 546  
 547  
 548  
 549  
 550

551 In Fig. 12, the spatial evolution of skewness, asymmetry, and kurtosis of  $ax(z_0)$  are shown. It is seen that the simulated results are in very good agreement with the measured  
 552 results. The evolution trends of these parameters are evidently different from those for  $\eta$  and  $u(z_0)$ , especially after the up-slope: the local peak of the skewness appears much  
 553 closer to the end of the up-slope, and the asymmetry parameter is positive over the bar. It is speculated that the differences are related to the phase differences between horizontal  
 554 acceleration and horizontal velocity. The kurtosis increases rapidly after the shoal and remains at a high level over the bar and a short distance over the de-shoaling zone. The evolu-  
 555 tion trends of these parameters are in line with the indication  
 556  
 557  
 558  
 559  
 560  
 561  
 562  
 563

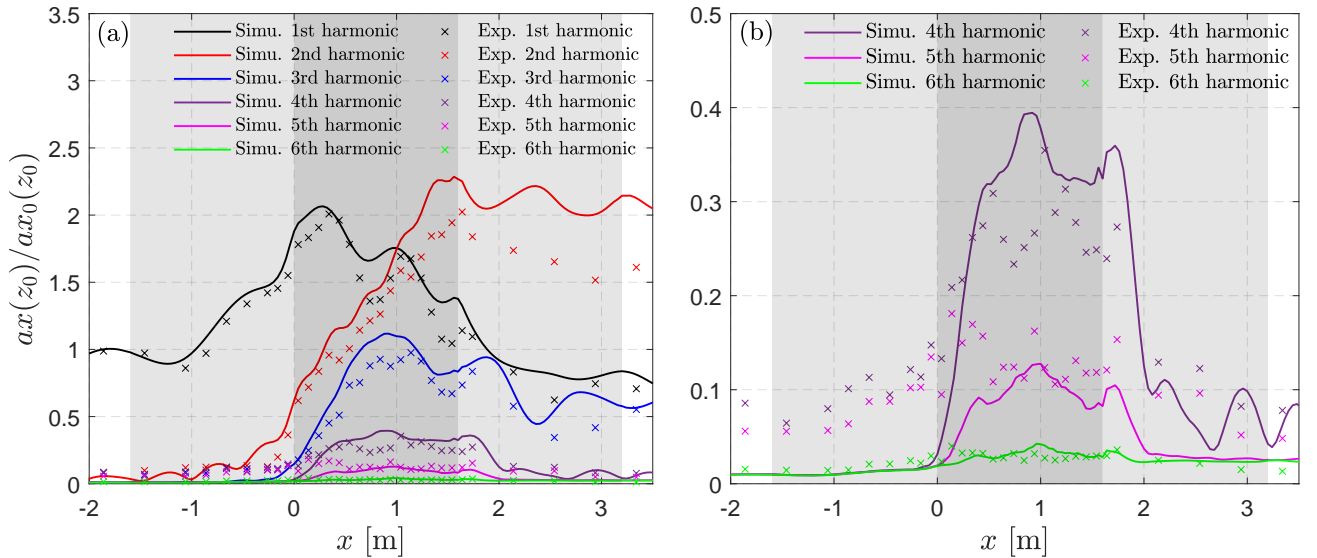
564 of stronger second- and higher-order harmonics observed in Fig. 11. It is stressed that the kurtosis after de-shoal is hard  
 565 to predict, and the W3D model performs well for that purpose, providing excellent prediction of not only the kurtosis  
 566 of FSE but also of the kinematics underneath.  
 567  
 568

## 5. Conclusion

569  
 570 In this study, new formulations of the particle kinematics, namely orbital velocities and accelerations, have been  
 571 developed in the FNPF wave model Whispers3D, which uses a basis of orthogonal Chebyshev polynomials of the first  
 572 kind to project the vertical structure of the velocity potential. With the potential expressed (and approximated) in a poly-  
 573 nomial form as given by eq. (8) where the main unknowns are then the  $a_n$  coefficients ( $n = 0, 1, \dots, N_T$ ), the expres-  
 574 sions of particle kinematics could be derived explicitly. The estimation of horizontal and vertical velocities involves the  
 575 spatial derivation of the potential in the corresponding direction, which can be obtained either analytically or with  
 576 the FD method without additional information in the time domain. However, the estimation of particle accelerations  
 577 involves the time derivative of the velocity components. In the model, this requires the computation of the time deriva-  
 578 tives of  $a_n$  coefficients, which can be obtained by using either a backward (in time) FD scheme in the course of the simu-  
 579 lation or a centered FD scheme after the completion of the run. The results shown here are obtained with a four-point  
 580 backward FD scheme (i.e. using the value at the current time plus the ones at the three previous time steps). The accuracy  
 581 and efficiency in the computation of the particle kinemat-  
 582



**Figure 10:** Comparison of the temporal profiles of the horizontal acceleration  $ax(z_0)$ , between the time-derivative of the measured horizontal velocity and the computed acceleration given by the model at 6 locations for the nonlinear regular wave experiment of Lawrence et al. (2021a).

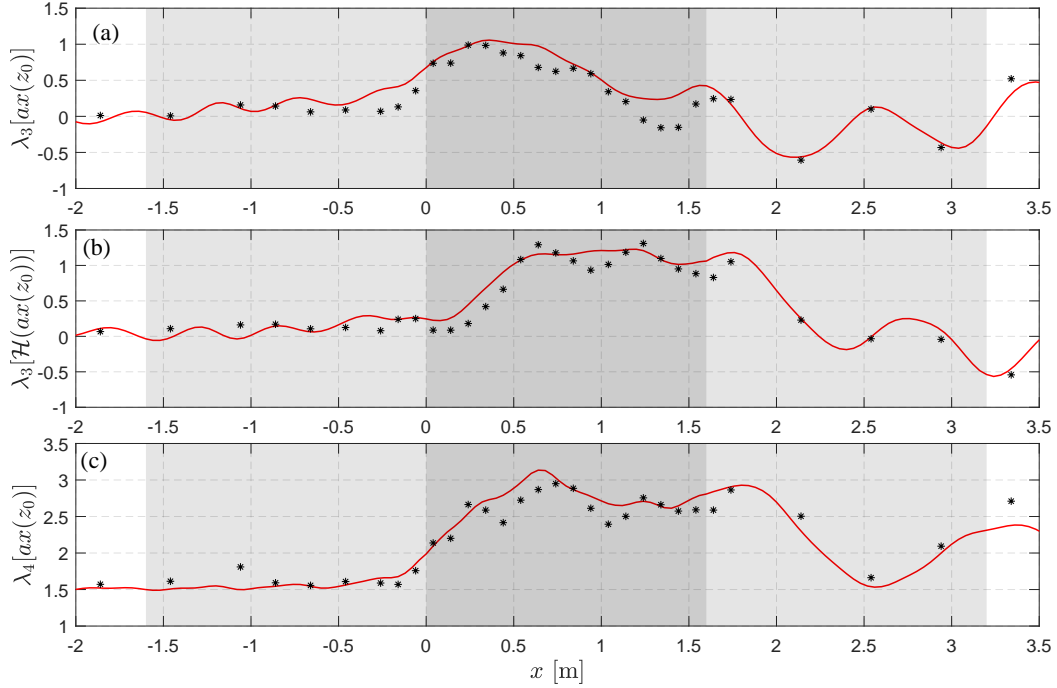


**Figure 11:** (a) Spatial evolution of normalized measured and computed amplitudes of the first 6 harmonics of the horizontal acceleration  $ax(z_0) = u_i(z_0)$  for the nonlinear regular wave experiment of Lawrence et al. (2021a). (b) Close-up view of the fourth to sixth harmonics, with a reduced extent of the vertical axis. The gray areas indicate the extent of the submerged bar.

593 ics are governed by the maximum degree of the Chebyshev  
594 polynomial  $N_T$  in eq. (8).

595 We have then demonstrated the capability and accuracy  
596 of the model to simulate kinematics beneath regular non-  
597 linear wave trains in either uniform or variable water depth

through comparisons with an analytical solution and exper- 598  
imental measurements. In the case of regular waves propa- 599  
gating over a flat bottom, a deep-water ( $kh = 2\pi$ ) and nearly- 600  
breaking ( $ka = 0.40$ ) wave condition is tested. With  $(\eta, \tilde{\phi})$  601  
calculated from the SF theory and imposed as initial condi- 602



**Figure 12:** Spatial evolution of statistical moments of the measured and computed horizontal acceleration  $ax(z_0) = u_t(z_0)$  for the nonlinear regular wave experiment of Lawrence et al. (2021a). The gray areas indicate the extent of the submerged bar.

603 tions on the free surface, the W3D model successfully computes  
 604 the kinematics in the whole fluid domain beneath the  
 605 free surface. The errors of computed FSE, velocities, and  
 606 accelerations in comparison to the reference SF solution oc-  
 607 cur mainly beneath the wave crest, yet remain very low (be-  
 608 low 0.1% for velocities and 1% for the accelerations) with  
 609  $N_T = 11$ . An even better agreement could be achieved with  
 610 a further increase of  $N_T$ , at the cost of an additional compu-  
 611 tational burden though. In the case of regular waves propa-  
 612 gating over an uneven bottom (submerged trapezoidal bar),  
 613 the experimental test reported in Lawrence et al. (2021b)  
 614 is reproduced with the W3D model. A Fourier analysis of  
 615 both measured and simulated time series shows a very good  
 616 to excellent agreement between simulated results and mea-  
 617 surements achieved for the amplitudes up to the sixth-order  
 618 super-harmonics of FSE, velocities, and accelerations. The  
 619 spatial evolution trends of statistical parameters describing  
 620 wave nonlinearity of the kinematics (skewness, asymmetry,  
 621 kurtosis) are also very well described by the model.

622 As a general conclusion, the W3D model is capable of  
 623 computing the kinematics beneath strongly nonlinear waves  
 624 very accurately under the framework of potential wave the-  
 625 ory. In comparison to other existing models, some advan-  
 626 tages of the W3D model can be summarized as follows:

- 627 1. It is a single-layer model that can handle nonlinear  
 628 waves in a broad range of relative water depth, whereas  
 629 a multi-layer approach is often required for higher-  
 630 order Boussinesq-type models aiming at a similar range  
 631 of application (see e.g. Liu and Fang, 2016; Fang et al.,  
 632 2022);
- 633 2. With the newly developed compact formulas, namely

634 eqs. (17)–(18) for the velocity components and eqs. (25)–  
 635 (26) for the acceleration components, explicit high-  
 636 order polynomial expressions are available to compute  
 637 the wave-induced kinematics at any point at or below  
 638 the free surface.

- 639 3. The model is free from any singularity issue when  
 640 computing the kinematics throughout the water col-  
 641 umn, which is different from other approaches as the  
 642 BEM for instance, in which singularities appear for  
 643 nodes located at the free surface (see e.g. Lefe et al.,  
 644 1980; Wang and Tsay, 2005).
- 645 4. It can describe kinematics beneath strongly nonlin-  
 646 ear waves with high accuracy. In contrast, tackling  
 647 the same question with the HOS modelling framework  
 648 can be of larger error close to the free surface, as ob-  
 649 served by Lawrence et al. (2021a) for instance, who  
 650 had to rely on a two-model approach, namely a HOS  
 651 model for simulating the wave field evolution and  
 652 a variational Boussinesq model to subsequently com-  
 653 pute wave kinematics.

654 In the companion Part II article (Zhang et al., 2024), we pro-  
 655 vide additional validation of the numerical model against ir-  
 656 regular wave experiments performed by Trulsen et al. (2020)  
 657 in which the same bottom profile as in section 4 was adopted.  
 658 The combination of measured and numerically simulated long  
 659 time-series of wave kinematics permits studying the statisti-  
 660 cal distributions of particle velocities and accelerations, and  
 661 proposing and validating a new statistical model of log-normal  
 662 type for these kinematics variables.

## 663 CRedit authorship contribution statement

664 **Michel Benoit:** Writing – Review & Editing, Writing –  
665 Original Draft, Supervision, Methodology, Conceptualiza-  
666 tion. **Jie Zhang:** Writing – Original Draft, Investigation,  
667 Funding acquisition, Formal analysis. **Yuxiang Ma:** Re-  
668 sources, Writing – Review & Editing, Supervision.

## 669 Declaration of competing interest

670 The authors declare that they have no known competing  
671 financial interests or personal relationships that could have  
672 appeared to influence the work reported in this paper.

## 673 Acknowledgments

674 This work was supported by the National Natural Sci-  
675 ence Foundation of China (Grant No. 52101301), and the  
676 China Postdoctoral Science Foundation (Grants No. 2023T160078,  
677 2021M690523). The authors would like to express their grati-  
678 tude to Christopher Lawrence, Karsten Trulsen, and Odin  
679 Gramstad for kindly sharing their experimental dataset in  
680 Lawrence et al. (2021a).

## 681 References

682 Aggarwal, A., Chella, M.A., Kamath, A., Bihs, H., Arntsen, Ø.A., 2016. Ir-  
683 regular wave forces on a large vertical circular cylinder. *Energy Procedia*  
684 94, 504–516. doi:10.1016/j.egypro.2016.09.223.  
685 Antuono, M., Colagrossi, A., Marrone, S., Lugni, C., 2011. Propagation of  
686 gravity waves through an SPH scheme with numerical diffusive terms.  
687 *Comput. Phys. Commun.* 182, 866–877. doi:10.1016/j.cpc.2010.12.012.  
688 Bassi, C., Bonaventura, L., Busto, S., Dumbser, M., 2020. A hyperbolic  
689 reformulation of the Serre-Green-Naghdi model for general bottom to-  
690 pographies. *Comput. Fluids* 212, 104716. doi:10.1016/j.compfluid.  
691 2020.104716.  
692 Belibassakis, K.A., Athanassoulis, G.A., 2011. A coupled-mode system  
693 with application to nonlinear water waves propagating in finite water  
694 depth and in variable bathymetry regions. *Coast. Eng.* 58, 337–350.  
695 doi:10.1016/j.coastaleng.2010.11.007.  
696 Benoit, M., Raoult, C., Yates, M.L., 2017. Analysis of the linear version of  
697 a highly dispersive potential water wave model using a spectral approach  
698 in the vertical. *Wave Motion* 74, 159–181. doi:10.1016/j.wavemoti.2017.  
699 07.002.  
700 Berkhoff, J., 1972. Mathematical models for simple harmonic linear water  
701 waves: wave diffraction and refraction. *Coast. Eng. Proc. (ICCE'1972)*  
702 1(13), 23. doi:10.9753/icce.v13.23.  
703 Bingham, H.B., Madsen, P.A., Fuhrman, D.R., 2009. Velocity potential  
704 formulations of highly accurate Boussinesq-type models. *Coast. Eng.*  
705 56, 467–478. doi:10.1016/j.coastaleng.2008.10.012.  
706 Boussinesq, J., 1872. Théorie des ondes et des remous qui se propagent le  
707 long d'un canal rectangulaire horizontal, en communiquant au liquide  
708 contenu dans ce canal des vitesses sensiblement pareilles de la surface  
709 au fond. *J. Math. Pures Appl.* 17, 55–108.  
710 Craig, W., Sulem, C., 1993. Numerical simulation of gravity waves. *J.*  
711 *Comput. Phys.* 108, 73–83. doi:10.1006/jcph.1993.1164.  
712 Dalrymple, R.A., Rogers, B.D., 2006. Numerical modeling of water  
713 waves with the SPH method. *Coast. Eng.* 53, 141–147. doi:10.1016/  
714 j.coastaleng.2005.10.004.  
715 Decorte, G., Toffoli, A., Lombaert, G., Monbaliu, J., 2021. On the use  
716 of a domain decomposition strategy in obtaining response statistics in  
717 non-Gaussian seas. *Fluids* 6, 28. doi:10.3390/fluids6010028.  
718 Deng, Y., Zhu, C., Wang, Z., 2023. A comparative study of wave kinematics  
719 and inline forces on vertical cylinders under Draupner-type freak waves.  
720 *Ocean Eng.* 288, 115959. doi:10.1016/j.oceaneng.2023.115959.

721 Didenkulova, E., Didenkulova, I., Medvedev, I., 2023. Freak wave events  
722 in 2005–2021: statistics and analysis of favourable wave and wind con-  
723 ditions. *Nat. Hazards Earth Syst. Sci.* 23, 1653–1663. doi:10.5194/  
724 nhess-23-1653-2023.  
725 Didenkulova, E.G., Pelinovsky, E.N., 2020. Freak waves in 2011–2018.  
726 *Dokl. Earth Sci.* 491, 187–190. doi:10.1134/s1028334x20030046.  
727 Didenkulova, I., Pelinovsky, E., 2016. On shallow water rogue wave for-  
728 mation in strongly inhomogeneous channels. *J. Phys. A: Math. Theor.*  
729 49, 194001. doi:10.1088/1751-8113/49/19/194001.  
730 Dommermuth, D., 2000. The initialization of nonlinear waves using  
731 an adjustment scheme. *Wave Motion* 32, 307–317. doi:10.1016/  
732 s0165-2125(00)00047-0.  
733 Dysthe, K.B., 1979. Note on a modification to the nonlinear Schrödinger  
734 equation for application to deep water waves. *Proc. R. Soc. A: Math.*  
735 *Phys. Eng.* 369, 105–114. doi:10.1098/rspa.1979.0154.  
736 Elfrink, B., Baldock, T., 2002. Hydrodynamics and sediment transport in  
737 the swash zone: a review and perspectives. *Coast. Eng.* 45, 149–167.  
738 doi:10.1016/s0378-3839(02)00032-7.  
739 Elgar, S., Guza, R.T., 1985. Observations of bispectra of shoaling surface  
740 gravity waves. *J. Fluid Mech.* 161, 425–448. doi:10.1017/  
741 S0022112085003007.  
742 Fang, K., Liu, Z., Wang, P., Wu, H., Sun, J., Yin, J., 2022. Modeling solitary  
743 wave propagation and transformation over complex bathymetries using  
744 a two-layer Boussinesq model. *Ocean Eng.* 265, 112549. doi:10.1016/  
745 j.oceaneng.2022.112549.  
746 Fenton, J.D., 1990. *The Sea - Vol. 9: Ocean Engineering Science, Part A.*  
747 Wiley. chapter 1. Nonlinear wave theories. pp. 3–25.  
748 Fochesato, C., Dias, F., 2006. A fast method for nonlinear three-  
749 dimensional free-surface waves. *Proc. R. Soc. A: Math. Phys. Eng.* 462,  
750 2715–2735. doi:10.1098/rspa.2006.1706.  
751 Fredsøe, J., Deigaard, R., 1992. *Mechanics of Coastal Sediment Trans-*  
752 *port. Advanced Series on Ocean Engineering: Vol. 3.* World Scientific.  
753 doi:10.1142/1546.  
754 Freilich, M.H., Guza, R.T., 1984. Nonlinear effects on shoaling surface  
755 gravity waves. *Philos. Trans. A: Math. Phys. Eng. Sci.* 311, 1–41. doi:10.  
756 1098/rsta.1984.0019.  
757 Gouin, M., Ducrozet, G., Ferrant, P., 2016. Development and validation of  
758 a non-linear spectral model for water waves over variable depth. *Eur. J.*  
759 *Mech. B Fluids* 57, 115–128. doi:10.1016/j.euromechflu.2015.12.004.  
760 Green, A.E., Laws, N., Naghdi, P.M., 1974. On the theory of water waves.  
761 *Proc. R. Soc. A: Math. Phys. Eng.* 338, 43–55. doi:10.1098/rspa.1974.  
762 0072.  
763 Grilli, S.T., Skourup, J., Svendsen, I.A., 1989. An efficient boundary element  
764 method for nonlinear water waves. *Eng. Anal. Boundary Elem.* 6,  
765 97–107. doi:10.1016/0955-7997(89)90005-2.  
766 Harris, J.C., Dombre, E., Benoit, M., Grilli, S.T., 2014. Fast integral equa-  
767 tion methods for fully nonlinear water wave modeling, in: *Proc. 24th Int.*  
768 *Ocean Polar Eng. Conf. (ISOPE'2014)*, Busan, Korea, 15–20 June 2014,  
769 pp. ISOPE–I–14–449.  
770 Hasimoto, H., Ono, H., 1972. Nonlinear modulation of gravity waves. *J.*  
771 *Phys. Soc. Japan* 33, 805–811. doi:10.1143/JPSJ.33.805.  
772 Higuera, P., Lara, J.L., Losada, I.J., 2013. Simulating coastal engineering  
773 processes with OpenFOAM®. *Coast. Eng.* 71, 119–134. doi:10.1016/  
774 j.coastaleng.2012.06.002.  
775 Jacobsen, N.G., van Gent, M.R., Wolters, G., 2015. Numerical analysis  
776 of the interaction of irregular waves with two dimensional permeable  
777 coastal structures. *Coast. Eng.* 102, 13–29. doi:10.1016/j.coastaleng.  
778 2015.05.004.  
779 Kriebel, D.L., 1998. Nonlinear wave interaction with a vertical circular  
780 cylinder: wave forces. *Ocean Eng.* 7, 597–605.  
781 Lafe, O.E., Liu, P.L.F., Liggett, J.A., Cheng, A.H.D., Montes, J.S., 1980.  
782 Singularities in Darcy flow through porous media. *J. Hydraul. Div.* 106,  
783 977–997. doi:10.1061/jyceaj.0005457.  
784 Larsen, B.E., Fuhrman, D.T., Roenby, J., 2019. Performance of interFoam  
785 on the simulation of progressive waves. *Coast. Eng. J.* 61, 380–400.  
786 doi:10.1080/21664250.2019.1609713.  
787 Lawrence, C., Gramstad, O., Trulsen, K., 2021a. Variational Boussinesq  
788 model for kinematics calculation of surface gravity waves over

- 789 bathymetry. *Wave Motion* 100, 102665. doi:10.1016/j.wavemoti.2020.  
790 102665.
- 791 Lawrence, C., Trulsen, K., Gramstad, O., 2021b. Statistical properties of  
792 wave kinematics in long-crested irregular waves propagating over non-  
793 uniform bathymetry. *Phys. Fluids* 33, 046601. doi:10.1063/5.0047643.
- 794 Li, Z., Tang, T., Li, Y., Draycott, S., van den Bremer, T.S., Adcock, T.A.A.,  
795 2023. Wave loads on ocean infrastructure increase as a result of waves  
796 passing over abrupt depth transitions. *J. Ocean Eng. Mar. Energy* 9,  
797 309–317. doi:10.1007/s40722-022-00269-4.
- 798 Liu, Z., Fang, K., 2016. A new two-layer Boussinesq model for coastal  
799 waves from deep to shallow water: Derivation and analysis. *Wave Mo-  
800 tion* 67, 1–14. doi:10.1016/j.wavemoti.2016.07.002.
- 801 Madsen, P.A., Fuhrman, D.R., Wang, B., 2006. A Boussinesq-type method  
802 for fully nonlinear waves interacting with a rapidly varying bathymetry.  
803 *Coast. Eng.* 53, 487–504. doi:10.1016/j.coastaleng.2005.11.002.
- 804 Madsen, P.A., Schäffer, H.A., 1998. Higher-order Boussinesq-type equa-  
805 tions for surface gravity waves: derivation and analysis. *Philos. Trans.  
806 Royal Soc. A* 356, 3123–3181. doi:10.1098/rsta.1998.0309.
- 807 Papoutsellis, C.E., Yates, M.L., Simon, B., Benoit, M., 2019. Modelling of  
808 depth-induced wave breaking in a fully nonlinear free-surface potential  
809 flow model. *Coast. Eng.* 154, 103579. doi:10.1016/j.coastaleng.2019.  
810 103579.
- 811 Paulsen, B.T., Bredmose, H., Bingham, H.B., 2014. An efficient domain  
812 decomposition strategy for wave loads on surface piercing circular cylin-  
813 ders. *Coast. Eng.* 86, 57–76. doi:10.1016/j.coastaleng.2014.01.006.
- 814 Porter, D., 2003. The mild-slope equations. *J. Fluid Mech.* 494, 51–63.  
815 doi:10.1017/s0022112003005846.
- 816 Raoult, C., Benoit, M., Yates, M.L., 2016. Validation of a fully nonlinear  
817 and dispersive wave model with laboratory non-breaking experiments.  
818 *Coast. Eng.* 114, 194–207. doi:10.1016/j.coastaleng.2016.04.003.
- 819 Raoult, C., Benoit, M., Yates, M.L., 2019. Development and validation of  
820 a 3D RBF-spectral model for coastal wave simulation. *J. Comp. Phys.*  
821 378, 278–302. doi:10.1016/j.jcp.2018.11.002.
- 822 Shi, J., Feng, X., Toumi, R., Zhang, C., Hodges, K., Tao, A., Zhang, W.,  
823 Zheng, J., 2024. Global increase in tropical cyclone ocean surface waves.  
824 *Nat. Commun.* 15, 174. doi:10.1038/s41467-023-43532-4.
- 825 Simon, B., Papoutsellis, C.E., Benoit, M., Yates, M.L., 2019. Comparing  
826 methods of modeling depth-induced breaking of irregular waves with a  
827 fully nonlinear potential flow approach. *J. Ocean Eng. Mar. Energy* 5,  
828 365–383. doi:10.1007/s40722-019-00154-7.
- 829 Stansberg, C.T., Huse, E., Krokstad, J.R., Lehn, E., 1995. Experimental  
830 study of non-linear loads on vertical cylinders in steep random waves,  
831 in: *Proc. 5th Int. Ocean Polar Eng. Conf. (ISOPE'1995)*, The Hague,  
832 The Netherlands, 11–16 June 1995, pp. ISOPE-I-95–013.
- 833 Tian, Y., Sato, S., 2008. A numerical model on the interaction between  
834 nearshore nonlinear waves and strong currents. *Coast. Eng. J.* 50, 369–  
835 395. doi:10.1142/s0578563408001879.
- 836 Trulsen, K., Raustøl, A., Jorde, S., Rye, L.B., 2020. Extreme wave statistics  
837 of long-crested irregular waves over a shoal. *J. Fluid Mech.* 882, R2.  
838 doi:10.1017/jfm.2019.861.
- 839 Vested, M.H., Carstensen, S., Christensen, E.D., 2020. Experimental study  
840 of wave kinematics and wave load distribution on a vertical circular  
841 cylinder. *Coast. Eng.* 157, 103660. doi:10.1016/j.coastaleng.2020.  
842 103660.
- 843 Wang, J., Tsay, T.K., 2005. Analytical evaluation and application of the  
844 singularities in boundary element method. *Eng. Anal. Bound. Elem.* 29,  
845 241–256. doi:10.1016/j.enganabound.2004.12.008.
- 846 Wang, Z., Zou, Q., Reeve, D., 2009. Simulation of spilling breaking waves  
847 using a two phase flow CFD model. *Comput. Fluids* 38, 1995–2005.  
848 doi:10.1016/j.compfluid.2009.06.006.
- 849 Wilson, J.F., 2002. *Dynamics of Offshore Structures*. Wiley.
- 850 Yates, M.L., Benoit, M., 2015. Accuracy and efficiency of two numerical  
851 methods of solving the potential flow problem for highly nonlinear and  
852 dispersive water waves. *Int. J. Numer. Meth. Fluids* 77, 616–640. doi:10.  
853 1002/flid.3992.
- 854 Zakharov, V.E., 1968. Stability of periodic waves of finite amplitude on  
855 the surface of a deep fluid. *J. Appl. Mech. Tech. Phys.* 9, 190–194.  
856 doi:10.1007/bf00913182.
- Zelt, J.A., Gudmestad, O.T., Skjelbreia, J.E., 1995. Fluid accelerations  
under irregular waves. *Appl. Ocean Res.* 17, 43–54. doi:10.1016/  
0141-1187(94)00019-j.
- Zhang, J., Benoit, M., 2021. Wave–bottom interaction and extreme wave  
statistics due to shoaling and de-shoaling of irregular long-crested wave  
trains over steep seabed changes. *J. Fluid Mech.* 912, A28. doi:10.1017/  
jfm.2020.1125.
- Zhang, J., Benoit, M., Ma, Y., 2022. Equilibration process of out-of-  
equilibrium sea-states induced by strong depth variation: Evolution of  
coastal wave spectrum and representative parameters. *Coast. Eng.* 174,  
104099. doi:10.1016/j.coastaleng.2022.104099.
- Zhang, J., Ma, Y., Benoit, M., 2024. Kinematics of nonlinear waves over  
variable bathymetry. Part II: Statistical distributions of orbital veloci-  
ties and accelerations in irregular long-crested seas. *Coast. Eng.* 193,  
104589.